The Poincare group, the Dirac monopole and photon localisation

1981 J. Phys. A: Math. Gen. 14 L73

(http://iopscience.iop.org/0305-4470/14/4/001)

J. Phys. A: Math. Gen. 14 (1981) L73–L74. Printed in Great Britain

LETTER TO THE EDITOR

The Poincaré group, the Dirac monopole and photon localisation

H Bacry

Centre de Physique Theorique, CNRS, Case 907, 13288 Marseille, Cedex 2, France

Received 20 January 1981

Abstract. The Dirac quantisation of a magnetic monopole is readily derived from the Lie algebra of the Poincaré group (taken as a dynamical group). The representation can be reinterpreted in order to define a position operator for massless particles.

We intend to show that the Poincaré group acts on the space of states of an electrically charged particle moving in the field of a Dirac magnetic monopole. The quantisation of the magnetic charge readily follows from what we know of the Poincaré group representations.

Many attempts have been made in the past for a group theoretical approach to the problem (Peres 1968, Lipkin *et al* 1969, Peshkin 1971). The most successful one was that of Lipkin *et al* (1969) who used a representation of the Euclidean group. Here we enlarge this group to the Poincaré group. Then no calculation is needed.

We consider a particle of charge e moving in the field $\boldsymbol{B}$ created by a magnetic monopole g. Due to the rotational symmetry of the problem, we can use the Noether theorem to get the corresponding conserved quantity (angular momentum)

$$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{\pi} - \lambda \boldsymbol{r}/r \tag{1}$$

where

$$\boldsymbol{\pi} = \boldsymbol{p} - e\boldsymbol{A}$$

$$\boldsymbol{B} = \text{curl}\,\boldsymbol{A} = g\boldsymbol{r}/r^3$$

$$\lambda = eg.$$

We now consider the operators

$$\boldsymbol{p} = \boldsymbol{r} \tag{2}$$

$$\boldsymbol{K} = \tfrac{1}{2}(r\boldsymbol{\pi} + \boldsymbol{\pi}r) \tag{3}$$

$$H = r. \tag{4}$$

It is a simple matter to verify that the operators $\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{P}, H$ span the Lie algebra of the Poincaré group. In fact:

$$[J_i, J_j] = \mathrm{i}\varepsilon_{ijk}J_k \qquad [J_i, P_j] = \mathrm{i}\varepsilon_{ijk}P_k$$

$$[J_i, K_j] = \mathrm{i}\varepsilon_{ijk}K_n \qquad [K_i, P_j] = \mathrm{i}\delta_{ij}H$$

$$[K_i, K_j] = \mathrm{i}\varepsilon_{ijk}J_k \qquad [K_i, H] = \mathrm{i}P_i$$

(all other commutators vanishing).

Obviously, since $H^2-P^2=0$ and $\lambda = \boldsymbol{J}\cdot\boldsymbol{P}/H$, we know that we have a representation of zero 'mass' and of 'helicity' $\lambda = eg$. Since the representation is unitary, the quantity eg is necessary a multiple of $\frac{1}{2}$.

We can now reinterpret this representation by making the following canonical transformation:

$$\boldsymbol{r}\to-\boldsymbol{p}$$

$$\boldsymbol{p}\to\boldsymbol{r}.$$

Equations (1)–(4) now read

$$\boldsymbol{J}=\boldsymbol{R}\times\boldsymbol{p}+\lambda\boldsymbol{p}/p \tag{5}$$

$$\boldsymbol{K}=p\boldsymbol{R} \tag{6}$$

$$\boldsymbol{P}=\boldsymbol{p} \tag{7}$$

$$H=p \tag{8}$$

with

$$\boldsymbol{R}=\boldsymbol{r}-e\boldsymbol{A}(-\boldsymbol{p}) \tag{9}$$

where $\boldsymbol{A}(-\boldsymbol{p})$ is the Fourier transform of $\boldsymbol{A}(\boldsymbol{r})$. In formulae (5)–(8), the operator $\boldsymbol{R}$ looks like a position operator for a massless particle with helicity λ. This seems to be in contradiction with the non-localisability of the photon. In fact, the components of $\boldsymbol{R}$ are not commuting. Rather

$$[R_i, R_j]=-\mathrm{i}\lambda\varepsilon_{ijk}p_k/p^3. \tag{10}$$

If we remember that there exists already a situation in physics where components of a position are not commuting, we could perhaps adopt $\boldsymbol{R}$ as a vector position for a massless particle. The situation we are referring to is the one of a charged particle moving in a plane homogeneous magnetic field. The classical trajectory is circular and the coordinates of the centre of the trajectory have non-commuting quantum counterparts (Landau and Lifshitz). In the present situation, we have the interesting fact that if we consider photons of higher and higher energy, the right-hand side of (10) becomes smaller and smaller and photons become more and more localisable. This last remark seems to us a good argument in favour of $\boldsymbol{R}$ as a good position operator.

The author is grateful to Professors A Peres and J Zak for their interest in the present work and to Professor Peshkin for his correspondence.

References

Landau L D and Lifshitz E M *Quantum Mechanics* **111** (Oxford: Pergamon)
Lipkin H, Weisberger W I and Peshkin M 1969 *Ann. Phys.* **53** 203
Peres A 1968 *Phys. Rev.* **167** 1449
Peshkin M 1971 *Ann. Phys.* **66** 542